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Integration of the Boltzmann Equation in the Relaxation Time Approximation

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We integrate by very simple means the Boltzmann equation in the relaxation time approximation. Our result improves on the solution previously found by Chambers, which does not take into account initial conditions.

KEY WORDS: Integration; Boltzmann equation; relaxation time approximation.

1. INTRODUCTION

In this note we integrate the linearized Boltzmann equation in the relaxation time approximation

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{F(x)}{m} \cdot \frac{\partial f}{\partial v} = -\frac{f - f^{(0)}}{\tau(x,v)}$$

$$\lim_{t \to 0} f(x,v,t) = \rho(x,v) \tag{1.1}$$

where (x,v) is in $\mathbb{R}^6$, dots indicate scalar products, and $\partial/\partial x$, $\partial/\partial v$ denote the usual gradients. Of course we assume that the force field $F: R^3 \to R^3$ is such that $dx/dt = v$ and $dv/dt = F(x)/m$ has global, unique solutions. Our result improves Chambers' results.[1,2]

Our method is an application of rather standard semigroup analysis and it is indeed a particularization (but independent of it) of more powerful methods for obtaining a probabilistic representation for a class of linear integrodifferential, nonhomogeneous, Cauchy boundary value problems.

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See Ref. 3 where also references to the real masters are given. Anyway, our technique seems not to have been applied and Chambers' method has made it to standard reference works.[4–6]

We obtain two different results depending on whether the $f^{(0)}$ in (1.1) satisfies either of

$$\begin{aligned} \frac{\partial f^{(0)}}{\partial t} + v \cdot \frac{\partial f^{(0)}}{\partial x} + \frac{F(x)}{m} \cdot \frac{\partial f^{(0)}}{\partial v} &= 0 \\ v \frac{\partial f^{(0)}}{\partial x} + \frac{F(x)}{m} \frac{\partial f^{(0)}}{\partial v} &= 0 \end{aligned} \tag{1.2}$$

or none of them. This distinction does not affect the method of solution but makes the difference with Ref. 1 (and users) more apparent.

In Section 2 we obtain the general solution to (1.1), in Section 3 we make some general comments on the solution, and we end with a simple application.

2. THE SOLUTION

Let us rewrite Eq. (1.1) as

$$\begin{aligned} \frac{\partial g}{\partial t} + v \cdot \frac{\partial g}{\partial x} + \frac{F(x)}{m} \cdot \frac{\partial g}{\partial v} &= -h(x,v,t) \\ \lim_{t\to 0} g(x,v,t) &= \rho(x,v) \end{aligned} \tag{2.1}$$

where

$$g(x,v,t) = e(t)f(x,v,t), \qquad h(x,v,t) = \frac{-e(t)f^{(0)}(x,v,t)}{\tau}$$

and

$$e(t) = \exp \int_0^t ds/\tau\left[\phi_{-s}(x,v)\right]$$

with $\phi_t(x,v)$ defined below. Observe that the differential operator on the left-hand side can be thought of as the generator of a flow on $\mathbb{R}^6 \times \mathbb{R}$ given by

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = \frac{F(x)}{m}, \qquad \frac{dt}{dt} = 1 \tag{2.2}$$

If $F: \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, we shall put $T_t F(x,v,s) = F(\phi_t(x,v), s+t)$, where $\phi_t: \mathbb{R}^6 \to \mathbb{R}^6$ is the flow associated to $dx/dt = v$, $dv/dt = F(x)/m$.

The T_t so defined defines a one-parameter group on any nice Banach space of functions on $\mathbb{R}^6 \times \mathbb{R}$. The infinitesimal generator of T_t is $G = \partial/\partial t + v \cdot \partial/\partial x + F(x)/m \cdot \partial/\partial v$, and the Green operator (resolvent) is $U^\alpha = \int_0^\alpha e^{-\alpha t} T_t \, dt$. It takes a straightforward calculation to verify that

$$(\alpha - G) U^\alpha = I \tag{2.3}$$

on appropriate domains (containing the continuous bounded functions). From this it follows that $g = U^\alpha h$ solves

$$(\alpha - G) g = h \tag{2.4}$$

and if we take the case where $\alpha = 0$ it follows that

$$g(x, v, t) = \int_0^\infty h(\phi_u, t + u) \, du \tag{2.5}$$

solves (2.4) with $\alpha = 0$, i.e., $Gg = -h$. Of course $h(x, v, t)$ must be such that (2.5) makes sense.

We have thus found a particular solution of the inhomogeneous equation (2.1), and in order to take care of the initial conditions we note that the solution of

$$\begin{gathered} Gg_1 = \frac{\partial g_1}{\partial t} + v \cdot \frac{\partial g_1}{\partial x} + \frac{F(x)}{m} \cdot \frac{\partial g_1}{\partial v} = 0 \\ g_1(x, v, 0) = \psi(x, v) \end{gathered} \tag{2.6}$$

is given by

$$g_1(x, v, t) = \psi[\phi_{-t}(x, v)] \tag{2.7}$$

Now, in order to solve (2.1) note that the value of $g(x, v, t)$ at $t = 0$ is

$$g(x, v, 0) = \int_0^\infty h[\phi_u(x, v), u] \, du$$

and that the solution of (2.6) with $\psi(x, v) = \rho(x, \rho) - g(x, v, 0)$ is

$$g_1(x, v, t) = \rho[\phi_{-t}(x, v)] - \int_0^\infty h[\phi_{u-t}(x, v), u] \, du$$

and therefore the complete solution of (2.1) is

$$g(x, v, t) = \rho(\phi_{-t}(x, v)) + \int_0^\infty h[\phi_u(x, v), t + u] \, du - \int_0^\infty h[\phi_{u-t}(x, v), u] \, du \tag{2.8}$$

In the formulas above we used the fact that $\phi_t \cdot \phi_s = \phi_{t+s}$ for any t, s. Replacing $h(x, v, t)$ by $-e(t) f^{(0)}(x, v, t)/\tau$, $g(x, v, t)$ by $e(t) f(x, v, t)$, we can write the solution to (1.1) as

$$\begin{aligned} f(x,v,t) &= e(t)^{-1}\rho\left[\phi_{-t}(x,v)\right] \\ &\quad -\int_0^\infty e(-u)f^{(0)}\left[\phi_u(x,v),u+t\right]\frac{du}{\tau\left[\phi_{-u}(x,v)\right]} \\ &\quad +\int_0^\infty e(t-u)^{-1}f^{(0)}\left[\phi_{u-t}(x,v),u\right]\frac{du}{\tau\left[\phi_{u-t}(x,v)\right]} \\ &= e(t)^{-1}\rho\left[\phi_t(x,v)\right] \\ &\quad +\int_0^t e(u)^{-1}f^{(0)}\left[\phi_{-u}(x,v),t-u\right]\frac{du}{\tau\left[\phi_{-u}(x,v)\right]} \end{aligned} \tag{2.9}$$

Let us now consider the case where $f^{(0)}$ satisfies either of the equations (1.2). Now, defining $g(x,v,t)=e(t)(f-f)^{(0)})$ and $\Psi(x,v)=\rho(x,v)-f^{(0)}(x,v,0)$, we are in case (2.1) with $h\equiv 0$ and a new initial condition which coincides with (2.6); therefore the solution is

$$f(x,v,t)=f^{(0)}(x,v,t)+e(t)^{-1}\Psi\left[\phi_{-t}(x,v)\right] \tag{2.10}$$

3. ANALYSIS OF THE SOLUTIONS

One obvious fact that follows from (2.9) and (2.10) is that for large t, the influence of the initial data disappears: it decays exponentially with rate τ^{-1}.

For f^0 satisying (1.2), the asymptotic behavior of f coincides with that of f^0. In the other case the analysis depends on the behavior for large t of f^0 as well as the behavior of $\phi_t(x,v)$ in the remote past. But there are some easy, general cases.

First when f^0 does not depend on t, then from (2.9) it follows that

$$f(x,v,t)\to\int_0^\infty e(u)^{-1}f^0\left[\phi_{-u}(x,v)\right]\frac{du}{\tau\left[\phi_{-u}(x,v)\right]} \qquad \text{as} \quad t\to\infty \tag{3.1}$$

Another possibility is that $f^0(x,v,t)\to f_e^0(x,v)$ as $t\to\infty$, uniformly in (x,v). In this case, for every fixed u we have $f^0[\phi_{-u}(x,v),t-u]\to f_e^0[\phi_{-u}(x,v)]$ as $t\to\infty$. Since the $f^0(x,v,t)$ and $f_e^0(x,v)$ are assumed to be such that the integrals involved are finite, we can pass to the limit under the integral sign and say

$$f(x,v,t)\to\int_0^\infty e(u)^{-1}f_e^0\left[\phi_{-u}(x,v)\right]du/\tau\left[\phi_{-u}(x,v)\right] \tag{3.2}$$

At this level of generality, little can be said about transport equations.

If $A(v)$ is some function of v, one is interested in the quantities

$$\langle A(v)\rangle(x,t) = \int A(v) f(x,v,t)\, dv$$

We see from (2.9), for example, that

$$\langle A(v)\rangle(x,t) = e(t)^{-1}\int A(v)\psi[\phi_{-t}(x,v)]\, dv$$

$$+ \int_0^t e^{-1}(u)\, du \int A(v) f^0[\phi_{-u}(x,v), t-u]\, du/\tau[\phi_{-u}(x,v)]$$

from which, and in case any of the conditions leading to (3.1) or (3.2) holds, we shall have

$$\langle A(v)\rangle(x,t) \to \int_0^\infty e(u)^{-1}\, du \int A(v) f_e^0[\phi_{-u}(x,v)]\, dv/\tau[\phi_{-u}(x,v)]$$

but little more can be said without being more specific about the $f^0(x,v,t)$.

4. COMMENTS AND A SIMPLE APPLICATION

The difference between our method and that of Refs. 1 and 2 is that we do not make assumptions on the change of $f(x,v,t)$ as a result of collisions but apply straightforward computations which enable us to get the result in a form in which initial conditions are taken into account. This would be relevant in problems where τ is large and transients may be significant. The result in Refs. 1 or 2 is only a particular solution to the inhomogeneous equation.

We should also stress that (1.1) is a gross simplification of the Boltzmann equation. A better linearization which takes into account collisions is described in Ref. 6. Also, some applications may require finite spatial volumes, in which case techniques like those in Ref. 3 should be invoked.

To finish let us consider a particle in an electric field E and a magnetic field B moving according to

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = c\left(E + v \wedge \frac{B}{c}\right)$$

where x, v, E, B are vectors and $\wedge$ denotes the standard vector product. Let Ω be an antisymmetric matrix such that $meB \wedge v/c = \Omega v$; then it is easy to see that

$$v(t) = U(t)v_0 + (e/m)R(t)E \tag{4.1}$$

where $U(t) = \exp(-\Omega t)$ and $R(t) = \int_0^t U(s)\, ds$. From (4.1) $x(t)$ is obtained by simple integration and

$$\phi_t(x,v) = x + R(t)v + e(m)\int_0^t R(s)\, ds\, E,\; U(t)v + (e/m)R(t)E$$

It follows from the last result of Section 3 that the equilibrium value of $\langle v\rangle(x,t)$ is given by

$$\langle v\rangle(x) = \int_0^\infty e(u)^{-1}\,du \int vf_e^0[\phi_{-u}(x,v)]\,dv/\tau[\phi_{-u}(x,v)]$$

If we assume that τ is a constant and f^0 is such that f_e^0 depends only on v, i.e., it is an unnormalized density and we interpret results as per unit volume, the last identity above can be written as

$$\langle v\rangle = \frac{1}{\tau}\int_0^\infty e^{-t/\tau}\,dt \int vf^0[U(-t)v + eR(t)E/m]\,dv \tag{4.2}$$

From the orthogonality of the matrix $U(t)$ it follows that

$$\int vf^0[U(-t)v + eR(t)E/m]\,dv = U(t)\int [v - eR(-t)E/m]f^0(v)\,dv$$

and if we assume furthermore that $\int vf^0(v)\,dv = 0$ and $\int f^0(v)\,dv = 1$ and use the fact that $U(t)R(-t) = -R(t)$, (4.1) can be rewritten as

$$\langle v\rangle = \frac{e}{m\tau}\left[\int_0^\infty e^{-t/\tau}R(t)\,dt\right]E = \frac{e\tau}{m}(1+\Omega\tau)^{-1}E$$

In the case $B = 0$ (or $\Omega = 0$) we obtain that

$$j = e\langle v\rangle = \frac{e^2\tau}{m}E$$

and when $B = (0,0,B_z)$ and $E = (E_x,0,0)$ another simple computation shows that

$$j_x = \frac{e^2\tau}{m}\left(1 + \frac{e^2B_z^2\tau^2}{m^2c^2}\right)^{-1}E_x$$

$$j_y = \frac{e^2\tau}{m}\left(1 + \frac{e^2B_z^2\tau^2}{m^2c^2}\right)^{-1}\left(\frac{eB_z\tau}{mc}E_x\right)$$

$$j_z = 0$$

from which the cases $eB_z\tau/mc \gg 1$ or $\ll 1$ can be obtained.

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